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Large N

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I review some of the things we have learned about large N gauge theories (and QCD_∞) from lattice calculations in recent years. I point to some open problems.

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1. Introduction

The fact that QCD has no obvious expansion parameter, (the coupling sets the length scale and cannot otherwise be independently varied) led 't Hooft to suggest [1] a parameter that is not obvious: $1/N$ where N is the number of colours. One thinks of the SU(N) theory as being the same as the SU(∞) theory up to corrections that are $O(1/N^2)$, or $O(1/N)$ in QCD_N. There are a number of ill-understood features of the strong interactions, such as the OZI rule, that become obvious and exact at $N=\infty$ [2]. However all these 'successes' require that $N=3$ should be 'close to' $N=\infty$; something that is not at all guaranteed. In particular, while the $N=\infty$ theory is simpler, it is unfortunately not simple enough to have been solved. There are however a number of things one can say on simple counting arguments [1, 2]. For example analysing perturbation theory to all orders tells us that to achieve a smooth $N=\infty$ limit one should keep the 't Hooft coupling, $\lambda = g^2N$, fixed as $N\to\infty$. Simple combinatorics (the colour singlet combination becomes vanishingly unlikely as $N\to\infty$) tells us that in the confining phase of that limit mesons and glueballs do not mix or decay, and indeed there are no colour singlet interactions at all. The lack of scattering makes it conceivable that integrability might play a role at $N=\infty$. Of course, all this depends on the $N=\infty$ theory having a confining phase – again something that is not guaranteed.

During the past decade, the large N limit has been at the center of one of the most exciting theoretical developments: the strong-weak coupling duality of AdS/CFT and its derivatives. (See [3] for a review.) As attempts are made to develop supergravity duals of non-conformal large-N field theories it is important to know something about the detailed properties of those theories to test the success of these attempts.

There are some simple questions here that lattice methods can help to answer. In this brief review I will say something about the following with some stress on open, accessible problems.

- Is large-N confining?
- Is SU(3) really close to SU(∞)?
- Mesons and QCD as $N\to\infty$.
- Space-time reduction at large-N.
- Hot SU(N) gauge theory.
- Interlacing θ-vacua.
- The spectrum of closed flux tubes in $D=3,4$.

The list of things I will not discuss is (unfortunately!) much longer. For example I will not discuss the work on large-N phase transitions that has been reviewed in Lattice 2005 and in Lattice 2007 [4]. Neither will I discuss numerical tests [5] of the remarkably successful analytic calculations by Karabali-Nair of string tensions in $D=2+1$ [6]. Nor will I review k-string tensions [7, 8], in $D=2+1$ or $D=3+1$, nor high-T domain wall tensions [9]; nor chiral symmetry breaking [10], and the role of topology [11], nor $\cdots$.

Throughout my talk I will try to point to problems which are accessible, interesting and are awaiting your involvement!

2. Is $N=\infty$ confining? Is $N=3$ close to $N=\infty$?

At $N=\infty$ decay widths are zero and we have a prefect OZI rule. This might explain why

in QCD we can have modest decay widths, e.g. $\Gamma_\rho/m_\rho \sim 1/5 \ll 1$, and a good OZI rule. At large N the explanation is essentially combinatoric: if the theory is confining, so that all states are colour singlets, we need the $q\bar{q}$ that pops out of the vacuum to have the right colour to make two colour singlet π when combined with the $q,\bar{q}$ in the ρ, and there is clearly a $1/N$ 'phase space' suppresssion there. If the theory is not confining any bound state can decay without any suppression into coloured states.

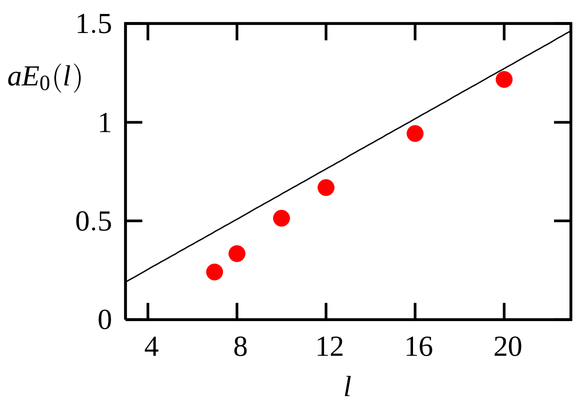
Is SU(∞) confining? A pedestrian but reliable approach to this question is to repeat for SU(4), SU(5), ... the kind of calculations that answered this question for SU(2) and SU(3). Let me show you an example for SU(6) [12].

We are on a $D=3+1$ hypertorus. Suppose its spatial dimensions are $l \times l_\perp^2$. Consider one unit of fundamental flux wrapped around the l-torus. If the theory is linearly confining this becomes a flux tube of length l and we expect its ground state energy to vary with l as

$$E_0(l) = \sigma l - c_{eff}(l)\frac{\pi}{6}\frac{D-2}{l} \stackrel{l\to\infty}{=} \sigma l - \frac{\pi}{6}\frac{D-2}{l} \tag{2.1}$$

where the correction term arises from the zero-point energies of the massless transverse oscillations (the Luscher correction) and we have assumed that the only massless modes along the flux tube are those of these stringy oscillations. This is the universality class of the simple free bosonic string, and it is important to determine numerically whether in fact $C_{eff}(l) \stackrel{l\to\infty}{=} 1$.

In Fig. 1 I show you what one obtains for SU(6). [12]. The approach to linearity at large l is evident. And we see that the deviations from linearity at smaller l, can be described by a $c_{eff}(l)$ that appears to approach unity at larger l. So it appears that the $N=\infty$ gauge theory is linearly confining and is in the bosonic string universality class.



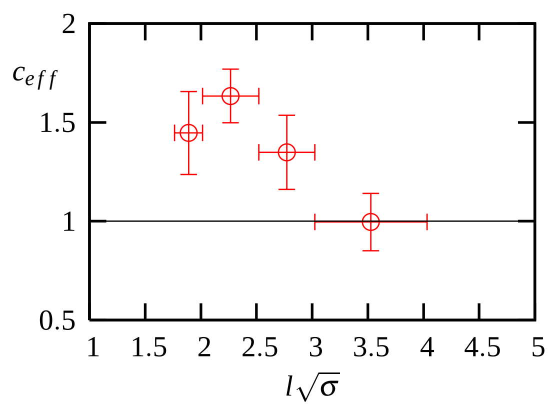


Figure 1: SU(6) : flux loop energy (left); effective Luscher correction (right).

This is a good start. However for large-N to be phenomenologically relevant, we need to show that for typical physical quantities the difference between SU(3) and SU(∞) is 'small'. So let us calculate the lightest $J^{PC}=0^{++},2^{++}$ glueball masses, express them in units of the simultaneously calculated string tension, and extrapolate the ratios to the continuum limit so as to obtain values of $m/\sqrt{\sigma}$. Now repeat this for various N. The leading large-N correction should be $O(1/N^2)$, so plot the resulting ratios [13, 8] against $1/N^2$ in Fig. 2. (For a very detailed comparison of SU(3) and SU(8) glueball spectra see [14].) We observe in Fig. 2 that the $O(1/N^2)$ corrections are indeed small, SU(3) appears to be 'close to' SU(∞), so large-N does appear to be physically relevant.

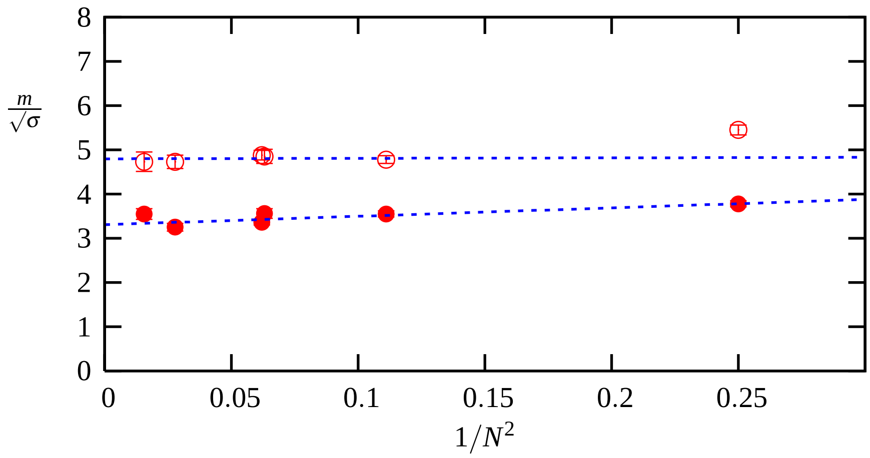


Figure 2: Continuum glueball masses in units of the string tension versus $1/N^2$.

3. QCD : $N = \infty$

Because quarks are in the fundamental, some corrections to QCD are $O(1/N)$ rather than $O(1/N^2)$, i.e. the size of the corrections might be closer to those of SU(2) than SU(3) in Fig. 2. Even so, we see from Fig. 2 that this is modest.

Since QCD_∞ is a (unitary) quenched theory, we can approach it through a sequence of quenched calculations of increasing N. These will generally have $O(1/N^2)$ corrections. If one calculates hadron masses at various N and extrapolates to $N = \infty$ at various fixed values of m_q, one can then do the usual chiral extrapolation in that limit.

A first step might be to do calculations not in the continuum limit but at some fixed small value of $a\sqrt{\sigma}$. And, if the calculation is not trying to be too precise, one can do chiral extrapolations at fixed N without worrying about the subtleties. In this way one gets the hadron spectrum at $N = \infty$, in units of $\sqrt{\sigma}$ or r_0. One can then compare it either to experiment or to full QCD lattice calculations, to see how large are the $O(1/N)$ corrections.

There have recently been two pioneering calculations of the latter kind [15, 16] that calculate $m_\rho/\sqrt{\sigma}$. (Also m_π, but that is traded for the physical m_q.) In Fig. 3 I show some figures from [16]. On the left is a chiral plot of m_ρ versus m_π^2 for $N \in [2,6]$ and on the right is the large N extrapolation of $m_\rho/\sqrt{\sigma}$. The N-dependence is clearly very weak, but this is expected for the quenched theory. More to the point, the $N = \infty$ chiral value is

$$\frac{m_\rho}{\sqrt{\sigma}} = 1.670(24) \sim 735\mathrm{MeV} \qquad ; \qquad \sqrt{\sigma} \simeq 440\mathrm{MeV} \tag{3.1}$$

which is within $\sim 35\mathrm{MeV} \sim \Gamma_\rho/4$ of the experimental value. These calculations [15, 16] thus provide explicit evidence that, as long hoped, QCD is close to QCD_∞.

Let me list a few of the many interesting questions that larger scale $N \to \infty$ calculations of this kind could address:

- Scalar mesons as $N \to \infty$: do the $\leq$ 1GeV states disappear?
- The scalar nonet and the place of lightest scalar glueball?
- Flavour singlet tensor and pseudoscalar mesons and glueballs?

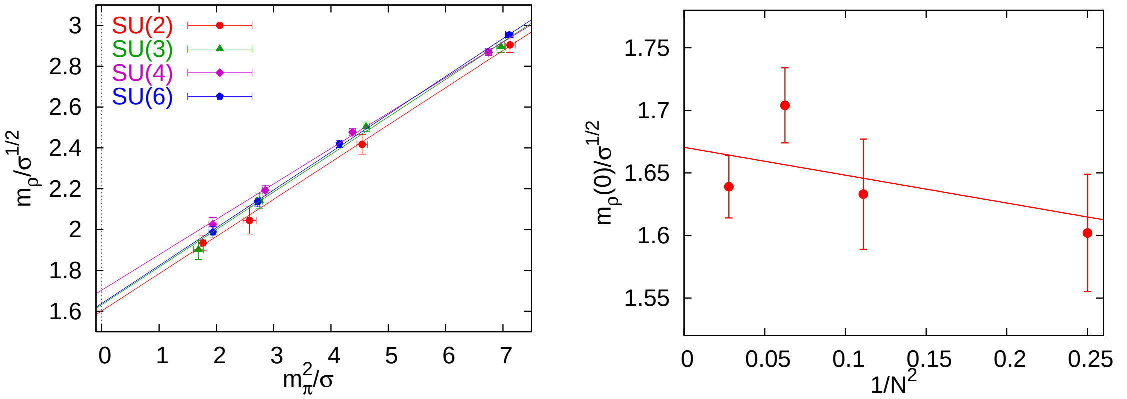


Figure 3: m_ρ versus m_π (left); m_ρ for $m_q = 0$ versus $1/N^2$ (right).

• Excited states stable $\longrightarrow$ Regge trajectories?
• Excited states stable $\longrightarrow$ clean meson excitation spectrum.
• SU($2n_f$) baryon (Dashen-Manohar) symmetry as $N \to 3$.
(This last might require full QCD_N – the next step.) Note that the suppression of decays and mixings as $N \uparrow$ might clarify many questions about the hadron spectrum that are obscured by large decay widths and mixings, and make such lattice calculations much cleaner and simpler at larger N than at $N = 3$. Answering the above questions would help answer some long-standing questions in hadron spectroscopy, e.g. are the $\leq$ 1GeV scalar mesons molecular? Do the three 0^{++} flavour singlets $\in 1.3-1.7$GeV arise from the scalar glueball mixing with two scalar nonet mesons? And this would give phenomenologists some concrete idea of how and where to look for the lightest tensor and pseudoscalar glueballs. Finally, the $N = \infty$ mesonic spectrum might show simple patterns that will provoke more theoretical understanding of that theory.

4. Calculating as $N \to \infty$: how much harder?

First some obvious counting. The pure gauge theory is dominated by the basic operation of multiplying two $N \times N$ matrices together, i.e. $O(N^3)$ operations. (The Cabibbo-Marinari heat-bath update of all SU(2) subgroups can be done in just $O(N^2)$ operations.) Current quenched QCD$_N$ calculations are, by contrast, dominated by the fermionic part where one is multiplying matrices times vectors, i.e. $O(N^2)$ operations.

Second, some less obvious counting. We calculate masses from connected correlators and in the pure gauge theory these are $O(1/N^2)$. Does this mean we need to increase the statistics to maintain the signal as $N \uparrow$? The answer is no: the statistical fluctuations are themselves related to higher order correlation functions and one can show [8] that they have the same $O(1/N^2)$ dependence: the error/signal ratio is independent of N. Indeed if one compares actual calculations of masses at different N one finds [8] very little N-variation in the error/signal ratio when one goes from SU(2) all the way to SU(8). One can apply the same analysis to meson propagators in quenched QCD. Just as the t-dependence of the error/signal ratio is very different for glueball and meson propagators (it grows exponentially for the former but is nearly constant for the latter) one finds a different

N-dependence: while it is $O(N^0)$ for the former one can show [16] that it is $O(1/N)$ for the latter. In practice the observed decrease [16] is closer to $O(1/\sqrt{N})$. So for QCD_N there is a gain.

In summary: the cost of pure gauge calculations is $\propto N^3$ while that of (quenched) QCD is $\propto N^{1.5}$ or even $\propto N$, unless one goes to very large N, where it will revert to N^3.

There are other less quantifiable gains at larger N. The existence of a first order strong-weak coupling transition for $N \geq 5$ means that one has some confidence about where one can start applying weak-coupling (Symanzik-type) analyses. The rapid disappearance of instantons with $\rho \sim O(a)$ as $N \uparrow$ [17] means that exceptional configurations (with their accompanying 'exceptional' fixes) should rapidly disappear. Also, the fact that finite volume corrections disappear as $N \to \infty$ means that we can work on smaller volumes. And the fact that excited states become stable means that they will be much easier to identify.

Pure gauge lattice calculations, with continuum limits and large N extrapolations of these, can and have been carried out on a few PC's. The quenched QCD calculations have used small clusters (or equivalents).

5. $\lambda = g^2 N$ fixed as $N \to \infty$?

Counting diagrams tells us that to have a smooth $N \to \infty$ limit we should keep $g^2 N$ constant. For a running coupling that means keeping $g^2 N(\mu)$ constant at constant $\mu/\sqrt{\sigma}$. That is to say, if we plot $g^2 N(\mu)$ against $\mu/\sqrt{\sigma}$ the result should not depend on N up to $1/N^2$ corrections. In Fig 4 we show how the (mean-field improved) bare coupling $g_I^2 N$ (a running coupling on the length scale a) varies with the scale, and it displays exactly such an N-independence [13]. In [18] it has been shown how the usual β-function with modest lattice corrections describes the running in Fig 4 and this allows the calculation of $\Lambda_{\overline{MS}}$ for all N.

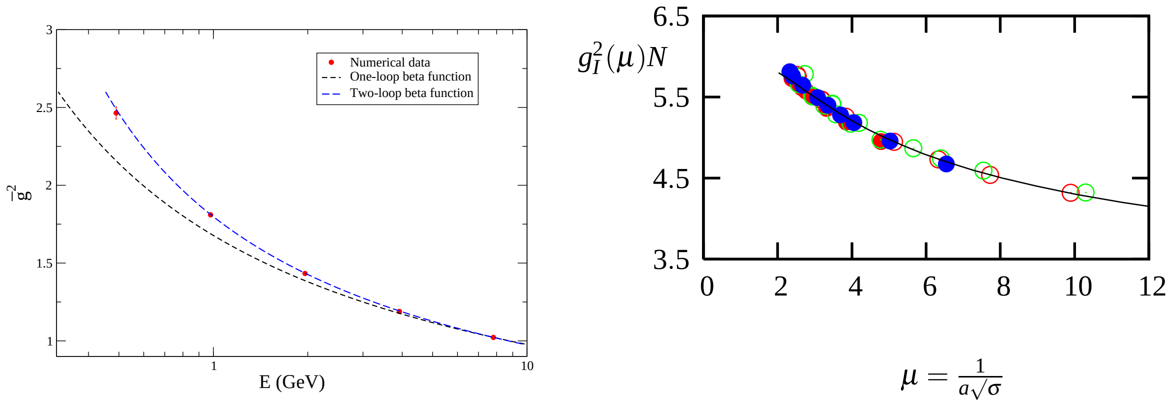


Figure 4: SF running coupling for SU(4) (left); bare 't Hooft running coupling for $N \in [2,8]$ (right).

Of course it would be nice to have a continuum β-function calculation at larger N. That has recently been provided in [19] for SU(4) using the Schrodinger functional definition. I reproduce it in Fig 4. Coupled with earlier SU(2) and SU(3) results, this leads to [19]

$$\frac{\Lambda_{\overline{MS}}}{\sqrt{\sigma}} = 0.528(40) + \frac{0.18(36)}{N^2} \tag{5.1}$$

which is entirely consistent with the values obtained in [18].

Question: does the $N=\infty$ SF coupling acquire non-perturbative jumps at the Narayanan-Neuberger [4] finite volume phase transitions?

6. Hot gauge theories at large-N

Calculations [20] for $N \in [2,8]$ show that the deconfining transition is first order for $N \geq 3$ with a latent heat of $O(1)$ when expressed in natural units. Now, at T_c the free energies of the gluon plasma and the confined ensembles are equal: $F_{gp} \stackrel{T=T_c}{=} F_{conf}$. So, since we expect $F_{gp} \propto N^2$ we are only interested in the $\propto N^2$ piece of F_{conf} at T_c when $N \to \infty$. The only piece in the confined phase that has this dependence on N is the vacuum energy density, which provides one conventional definition of the gluon condensate. (Without this we could expect $T_c \to 0$ as $N \to \infty$.) So if the gluon plasma was weakly coupled, we could use the usual Stefan-Boltzmann (SB) expression for F_{gp} and thus obtain T_c in terms of the vacuum energy density and the gluon condensate – which would be a very nice prediction! Unfortunately for us (fortunately for our AdS/CFT colleagues - see below) the plasma turns out to be strongly coupled near T_c.

The deconfining temperature is shown as a function of $1/N^2$ in Fig. 5 [20]. We see modest corrections to the $N=\infty$ limit: the fit is $T_c/\sqrt{\sigma} = 0.597(4) + 0.45(3)/N^2$. Remarkably this even fits SU(2), where the transition is second order.

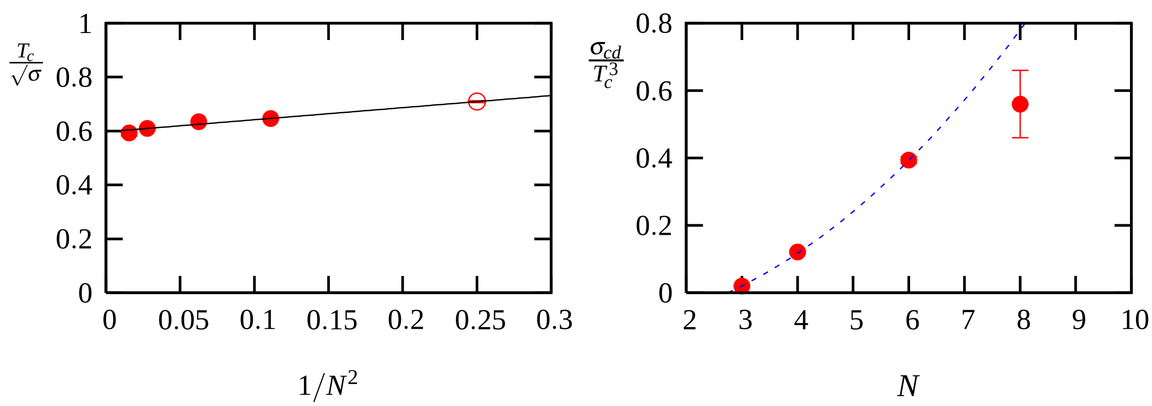


Figure 5: N-dependence of the deconfining temperature (left) and the interface tension (right).

We also show in Fig. 5 the interface tension [20] between the confined and deconfined phases. We see that the SU(3) value is strongly suppressed: indeed the fit shown is $\sigma_{cd}/T_c^3 = 0.0138N^2 - 0.104 \stackrel{N=3}{\simeq} 0.020$. This suppression is much stronger than the modest suppression of the SU(3) latent heat, and is presumably the real reason why that transition has looked so ‘weakly’ first order in the past.

One finds a very similar pattern of results in $D=2+1$ [21] except that now it is SU(4) that is weakly first order and, as expected, T_c is larger, $T_c/\sqrt{\sigma} = 0.903(3) + 0.88(5)/N^2$.

For quite a large range of T above T_c we have a non-trivial strongly coupled (quark-)gluon plasma, that is being probed by experiments at RHIC and (soon) LHC. Calculating the kinetic properties of this plasma has become the area of choice for gauge-gravity applications [22]: we

have a strongly coupled system, and finite T breaks the supersymmetry (fermions become massive, scalars are then unprotected and also become massive), so the resulting theory (possiby rescaling the number of degrees of freedom) may be 'not so different' from QCD. However one major issue is that gauge-gravity dualities tell us about $N = \infty$ not $N = 3$. So, is the observed strongly coupled plasma a feature of just SU(3), or does it survive as $N \to \infty$?

This answer is that it survives as $N \to \infty$ [23]. In Fig. 6 we see how the (lattice SB normalised) pressure varies with N [23]; clearly $SU(3) \simeq SU(\infty)$ for this quantity that provides one characteristic measure of strong coupling. For balance I also show $\Delta = \varepsilon - 3p$ which is a measure of the breaking of conformality: that also suvives large N. Nonetheless this provides an important piece of the motivation for believing AdS/CFT may be applicable to this physics.

The fact that the pressure 'anomaly' survives at $N = \infty$ has implications for the dynamics; for example, at $N = \infty$ it cannot be due to instantons, or to the survival of colour singlet hadrons.

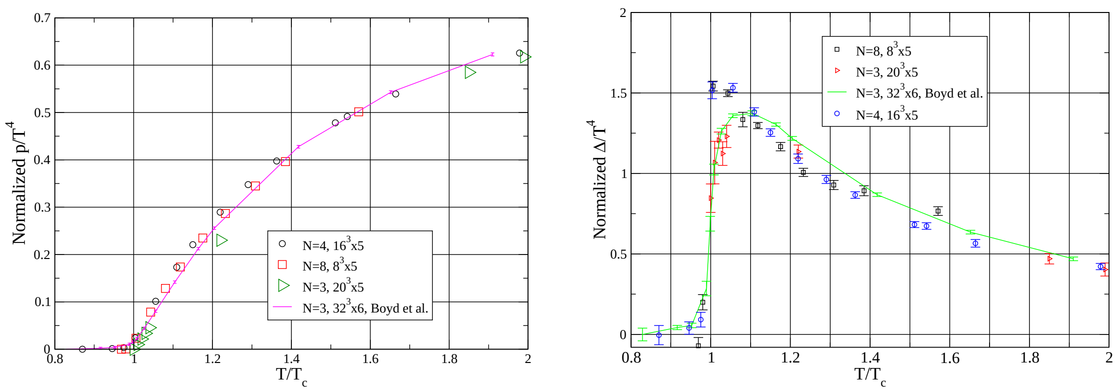


Figure 6: N-dependence of pressure (left), $\varepsilon - 3p$ (right) for $T > T_c$.

The critical result for motivating AdS/CFT for $T > T_c$ was the result for shear viscosity divided by entropy, $\eta/s = 1/4\pi$, and the fact that it is close to RHIC, and is 'universal' for any theory with a gauge/gravity dual. So in both the experimental and theoretical communities, a lattice calculation of η/s is eagerly awaited. As you have heard in the status report by Harvey Meyer [24] in his Plenary talk at this meeting, his large-scale lattice calculations of η/s look like being consistent both with experiment and with AdS/CFT.

It might be interesting to repeat these calculations of η/s for $D = 2+1$ gauge theories. This might tell us whether the theory has a gravity dual or not?

7. Space-time reduction at large N

At $N = \infty$ we have the factorisation of gauge-invariant operators: $\langle \mathrm{Tr}\Phi_a \mathrm{Tr}\Phi_b \rangle = \langle \mathrm{Tr}\Phi_a \rangle \langle \mathrm{Tr}\Phi_b \rangle$. This implies that there is a single gauge field (gauge orbit) that dominates the path integral. Since physics is translation invariant, this gauge field must also be translation invariant (for gauge invariant quantities). Thus one can imagine that the $N = \infty$ theory could be defined on any volume, even infinitesimal. On a lattice, this would be a single site: $L^4 \to 1^4$. This heuristic argument is made concrete in Eguchi-Kawai (EK) reduction [25] which tells us that at $N = \infty$ the Schwinger-Dyson

equations for Wilson loops on a 1^4 lattice are the same as for the $L = \infty$ lattice theory. This requires that the center symmetry $U_l \to zU_l$, $z \in Z_N$, which follows because the plaquette is $U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger$ on a 1^4 torus, is not spontaneously broken. To maintain this one has to impose twisted boundary conditions (TEK) [26]. In the mid-1980's there were a number of lattice calculations using in the TEK model that claimed to calculate physical quantities such as T_c at very large $N \sim O(64)$ [27].

Recently this has become a very active area again, and a number of long-standing beliefs have been demolished in the process.

Firstly, it was shown in [28] that at larger N the center symmetry is spontaneously broken in the TEK model, for an important range of bare couplings that increase as N increases. In [29] it was then shown that this range appears to increase as $N \to \infty$ in such a way as to probably prevent a planar continuum limit. Moroever even where the symmetry is not broken, the TEK model seems to be stuck in the wrong phase [28]. Typically the physically desired vacuum is only $O(N)$ below undesired vacua at tree-level, and $O(N^2)$ quantum fluctuations overwhelm this difference – and produce insurmountable barriers to tunnelling. So it appears that the only planar physics you can get from TEK is the physics of an infinitesimaly small volume [28] – and we hardly need lattice calculations for that!

An alternative to TEK, proposed around the same time, was quenched (in the replica sense) Eguchi-Kawai (QEK) [30]. Following the demise of the TEK model, there has recently been a renewed analysis of the QEK model in [31] which has found that, in a rather more subtle way, the center symmetry is broken here as well. So this route to planar space-time reduction fails as well.

That leaves two approaches. First one can deform the action so as to bias the system towards maintaining center symetry. This has been recently formulated in a precise way in [32]. This looks a promising route for reducing one or two space-time directions, but it might not be practical for going beyond that. It would be very interesting to see some numerical investigations of this idea, to see what the costs are in practice (and this is in progress [33]).

Finally there is the 'no tricks' approach to reduction [4]. Here you give up on complete space-time reduction, and instead go as far as you can using the fact that finite volume corrections disappear as $N \to \infty$, as long as you remain on a l^4 volume that remains in the confining phase, i.e. $l > 1/T_c$. This means working at large enough N that any $O(1/N^2)$ finite volume corrections are smaller than your statistical errors, i.e. very large N indeed. However if you want to do that anyway, then there is no extra cost. There has been a nice demonstration of how this can work in a recent calculation of the $D = 2+1$ string tension, $\lim_{N\to\infty} \sqrt{\sigma}/g^2$, in [34] performed for $N \in [21, 47]$, which agrees with conventional calculations [35, 5] extrapolating from $N \in [2, 8]$. The method applies equally well to fermions.

Time has forced me to summarise this area as a collection of sound bites. However space-time reduction touches on a number of deep and fascinating issues in field theory.

8. Interlaced θ-vacua

Usually we act as though confining gauge theories had just one vacuum. The reality, however, is much more interesting ...

Consider the gauge action with a θ term

$$S[g^2,\theta] = \frac{1}{4g^2}\int d^4x \mathrm{Tr} F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{16\pi^2}\int d^4x \varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr} F_{\mu\nu}F_{\rho\sigma} \tag{8.1}$$

Since

$$\frac{1}{16\pi^2}\int d^4x \varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr} F_{\mu\nu}F_{\rho\sigma} = Q = \text{integer} \tag{8.2}$$

we know that $\exp\{-S[\theta]\}$ and hence the vacuum energy density $E(\theta)$ are periodic in θ

$$E(\theta) = E(\theta + 2\pi) \qquad \forall N \tag{8.3}$$

On the other hand, we expect that for a smooth $N \to \infty$ limit, we need to factor N from S so that the couplings to keep fixed are $1/g^2N$, θ/N, ... i.e.

$$E(\theta) = N^2 h(\theta/N) \tag{8.4}$$

How do we reconcile these two apparently irreconciliable demands?

Witten's suggestion [36] is that $E(\theta)$ is a multi-branched function

$$E_k(\theta) = N^2 h\left(\frac{\theta + 2\pi k}{N}\right) \quad ; \quad E(\theta) = \min_k E_k(\theta) \tag{8.5}$$

so that $E(\theta) = E(\theta + 2\pi)$ while each $E_k(\theta)$ is periodic in $2\pi N$. See Fig 7 for $N = 10$.

Now, the domain wall tension between different 'k-vacua' is $O(N)$ so as $N \to \infty$ these will all become stable [36, 37, 38] and there are arguments this will happen already at modest N (=3?). So if we look at $\theta = 0$ in Fig 7, we see that we have a whole tower of nearly stable vacua that are above the true vacuum, and which will, at some other θ, themselves become the true vacua.

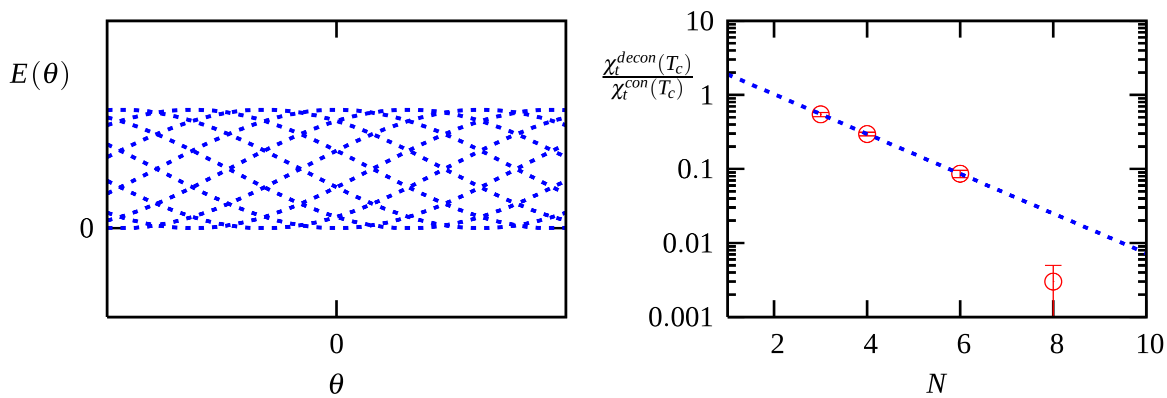


Figure 7: Interlaced θ-vacua for SU(10) (left); topological suseptibility across T_c (right).

There are some things we have already learned from the lattice.

• $T \leq T_c$: we expect N vacua at any given θ, and there is some nice lattice evidence for this scenario in [39] where these authors show that the periodicity of our usual vacuum is indeed much greater than the naive 2π, by the simple yet effective astuce of calculating high moments of Q.

• $T > T_c$: topological fluctuations disappears roughly exponentially in N [40, 41], as you can see

in the right hand plot [40] of Fig. 7. So in the deconfined phase there need be no interlaced vacua – just naive 2π periodicity in θ with exponentially small $E(\theta)$ variation.

So as $N \to \infty$ there are N stable vacua at $\theta = 0$. Now at large N and at $\theta = 0$, the lowest of these new vacua are close to their minima, and there we can use a quadratic approximation for $E_k(\theta)$, giving :

$$E_k(\theta = 0) = E_0(\theta = 0) + \frac{1}{2}\chi_t(2\pi k)^2 \qquad : \quad k \ll N \tag{8.6}$$

where $\chi_t = \langle Q^2 \rangle / V$ is the topological susceptibility. Thus the gap to the first of these vacua is

$$E_{k=1} - E_0 = 2\pi^2 \chi_t \sim (360 MeV)^4 \tag{8.7}$$

in contrast to the $E_0 \sim -O(N^2)$ vacuum energy. This is a modest gap that will be much smaller, at even moderate N, than, for example, the latent heat of the confinement-deconfinement transition, $L_h \sim 0.77 N^2 T_c^4$ [20].

These near-stable vacua should appear as quasistable in computer time when using local Monte Carlo updates and we should be able to find them! How might we identify such k-vacua?

Firstly, in a $k \neq 0$ vacuum $\langle Q \rangle \neq 0$ since these vacua are not CP-invariant, $k \stackrel{CP}{\to} N - k$, although they come in degenerate pairs and will therefore mix – although there is a large barrier. This signal will be tricky to make use of at large N, where transitions between sectors of different topological charge are suppressed exponentially in N (the famous $\propto \exp\{-8\pi N/(g^2(\rho)N)\}$ factor for small instantons) because for Q to change, an instanton has to become small before it shrinks out of a hypercube (or the reverse). So unless we work at a very coarse a, sequences of configurations will get locked into fixed Q so that $\langle Q \rangle$ becomes incalculable.

Secondly, we expect the string tension to decrease with increasing k, since the vacuum energy (the 'bag constant') increases. This might go as [38] $\sigma(k) \simeq \sigma(k=0)\cos\{\frac{2\pi k}{N}\}$. (Suggesting that the upper half of vacua, $N/4 \leq k \leq 3N/4$, are unstable.) To use this we need enough metastability to be able to acquire high statistics in this false (k-)vacuum.

Thirdly, since the vacuum energy increases with k, $E(k) = E(k=0) + O(k^2)$, perhaps such a state deconfines at a lower temperature: $T_c(k \neq 0) < T_c$?

So how do you produce such vacua? One possibility is to 'quench' in β across the strong-weak coupling bulk transition. Here the latent heat is $O(N^2)$ and one might hope to fall into one of these confining k-vacua more or less at random. I should add that I tried this a few years ago but could not make it work. (It might be because the strong coupling phase has a small correlation length, so the configuration after quenching is full of bubbles, and the usual vacuum bubble then grows and takes over.) Another, probably more promising possibility is to 'quench' in β from $T > T_c$ to low T.

Note that one of the fascinating aspects of these vacua is that they are almost certainly the pure gauge counterparts of the degenerate vacua in the confining phase of SU(N) $\mathcal{N} = 1$ SUSY. To be specific, start in the latter theory, where one has a set of massless adjoint fermions (gluinos). There is an anomaly and a spontaneous symmetry breaking that leads to a gluino condensate [42] whose phase is an element of the center and labels the degenerate vacua. Now make the gluino mass finite, $m_{qA} \neq 0$, thus explictly breaking the degeneracy. Now take $m_{qA} \to \infty$, so that the gluinos decouple and we are left with the pure gauge theory. The idea is that as we do this [37] the originally degenerate vacua become the k-vacua we have been discussing here. It would be of great interest,

to a much wider theory community than the one we usually address with lattice QCD, if we were to explicitly demonstrate such a scenario in a lattice calculation.

9. Flux tubes as strings

The idea that the strong interactions might be formulated as a string theory, is older than QCD. Indeed string theory began with the Veneziano model (late 60's) and for the first few years was being developed as a theory of the strong interactions. The fact that at large N diagrams can be naturally mapped to a sequence of manifolds that look like a perturbative expansion in string theory, and that at strong 't Hooft coupling the vertices become dense over these manifolds, suggests ('t Hooft, mid-70's) that at large-N, and particularly at strong coupling, the gauge theory might be a string theory. This is a limit that has recently been resuscitated in the gauge-gravity duality which provides a precise string description for certain large-N gauge theories at strong coupling (Maldacena, late 90's). And finally, the string-like character of the confining flux tube, and the fact that Wilson loops are natural variables for a confining gauge theory, also motivated the construction of effective string actions (Polyakov, late 70's).

The natural starting point for building up a stringy description is to investigate the properties of confining flux tubes. This has been an active research area from the earliest days of lattice calculations. As a result of this, it is now fairly clear that long flux tubes can be described by an effective string theory [43] that is in the universality class of the free bosonic string theory, i.e. Nambu-Goto in flat space-time. In recent years there has been a great deal of work in both $D = 3+1$ and $D = 2+1$ (which are, for these purposes, equally interesting). For example, comparing Wilson loops to Nambu-Goto (Caselle, Gliozzi, ...), SU(3) string spectrum (Kuti, ...), potentials in $D = 2+1$ both for SU(2) (Hari Dass, Majumdar, ...) and SU(5) (Meyer). Here I will briefly describe a current programme in this area that I have been involved in, with Barak Bringoltz and Andreas Athenodorou. It is for SU(N) gauge theories in $D = 2+1$ and it is currently being extended to $D = 3+1$.

If we just want to consider the properties of a confining flux tube, it is convenient to have a set-up without sources. So suppose we want a flux tube of length l that closes on itself. To ensure its stability, make it wind around a spatial torus chosen to have size l. The correlators will involve operators that are variants of smeared/blocked Polyakov loops that wind around the spatial (not temporal!) torus. Other lattice dimensions are made effectively ∞. There is a phase transition at $l_c \equiv 1/T_c \simeq 1.1/\sqrt{\sigma}$ such that for $l < l_c$ we have no confining flux tube. So we can study the spectrum of such flux tubes for all $l > l_c$ and compare them to simple string expectations.

We start with the ground state of the flux tube [44, 45]. In Fig. 8 I show the effective Luscher coefficent c_{eff} defined in eqn(2.1). On the left SU(5), open squares; on the right SU(2) and open circles. The approach to the free bosonic string value, $c_{eff} = 1$, looks unambiguous, and very similar for SU(2) and SU(5). The other sets of points shown are obtained by modifying the ground state energy of the Nambu-Goto string, i.e.

$$E_0(l) = \sigma l \left(1 - c_{eff}(l)\frac{\pi}{3}\frac{D-2}{\sqrt{\sigma}l^2}\right)^{\frac{1}{2}}. \tag{9.1}$$

where again $c_{eff}(l) = 1$ would be the Nambu-Goto value. As we see $c_{eff} = 1$ for almost all l, i.e. the free string expression describes flux tubes down to $l\sqrt{\sigma} \sim 1$ where they are presumably fat blobs, not slim tubes at all! Note that when we expand $E_0(l)$ in eqn(9.1) in powers of $1/\sqrt{\sigma}l^2$ we get the Luscher correction in eqn(2.1) as the first correction. So what we have learned here is that the higher order corrections are also Nambu-Goto to a very good approximation. In fact we now know theoretically [47] that the next term in the expansion is universal and the same as in Nambu-Goto. However the agreement we are seeing goes much further than that: if we fit with Nambu-Goto and include the first non-universal correction we find that its coefficient is unnaturally small, $C < O(0.1)$, when expressed in natural units.

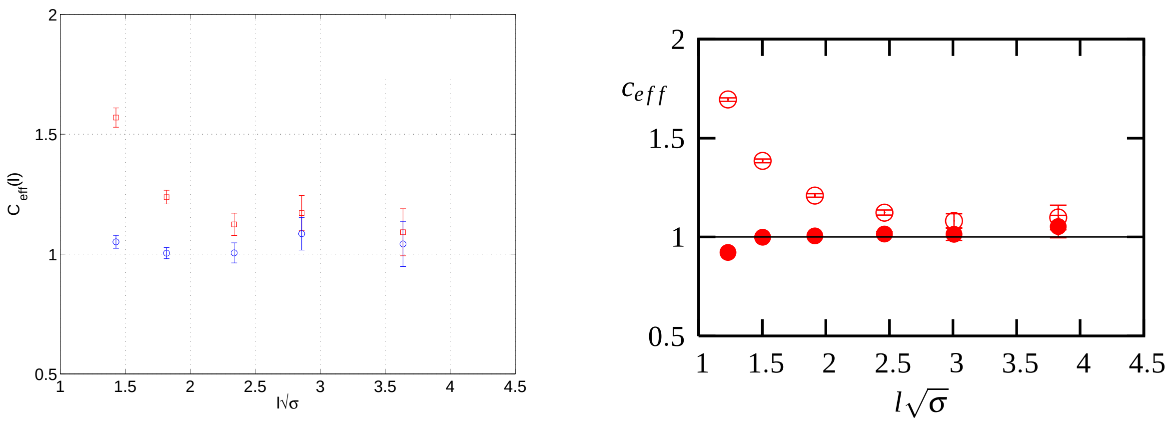


Figure 8: Effective Luscher and Nambu-Goto coefficients for SU(5) (left) and SU(2) (right).

To show how much further this goes, look at the low-lying flux loop spectrum in Fig. 9 [44]. (For zero momentum along the flux tube.) The solid lines are the predictions of Nambu-Goto. Essentially the excitations correspond to free massless 'phonons' travelling along the flux loop, with momenta that are quantised by the periodicity of the string. The ground and first excited state are predicted to have parity $P = +$ and the second excited level has two $P = +$ and two $P = -$ states that are degenerate. The numerical results in Fig. 9 reproduce precisely this pattern of degeneracies and quantum numbers. Note that the only parameter in Nambu-Goto is the string tension which is obtained from the ground state fit. The excited states are thus predictions with no free parameter at all. I invite you to be surprised that a flux tube of length $l\sqrt{\sigma} \sim 2$, surely a blob, has a first excited state that is precisely given by the oscillation of a thin string.

In Fig. 9 the long-dashed lines are what you get with just the first (Luscher) correction, and the short-dashed lines include the next universal correction. These are clearly no use for excited states in this region of l. The reason is that for most of this range, the expansion in $1/\sqrt{\sigma}l^2$ is divergent for these excited states. (As is the expansion of the Nambu-Goto expression.)

So the message is that the starting approximation one should use is the Nambu-Goto spectrum, rather than some large-l truncation thereof. It is interesting that $k = 2$ flux tubes, which are bound states of two $k = 1$ flux tubes, do show $O(1)$ corrections to Nambu-Goto, as one would expect in a long-distance effective string description [45, 46]. This contrast highlights the significance of what one sees here for the fundamental flux tube. This suggests that somehow even short blobby fundamental flux tubes know that they are really strings. This is not natural in an arbitrary 'Nielsen-

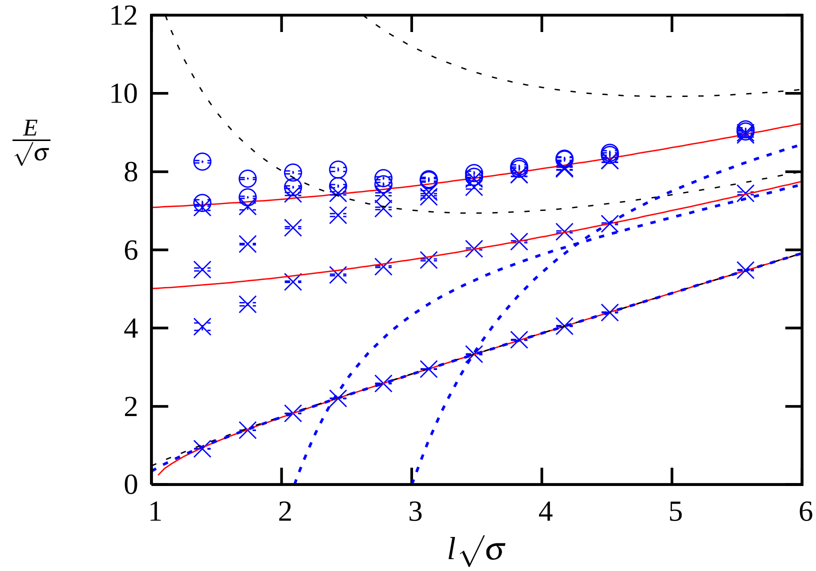


Figure 9: Flux loop spectrum in SU(3): solid lines Nambu-Goto, dashed lines see text.

Olesen vortex' type of picture. But it is what happens in gauge-gravity duals.

It is now of great interest to determine the qualitative features of the inter-phonon interactions, and to try and find non-string ('breathing'?) modes that should be there, even in a gauge-gravity dual picture, and which might give hints as to the appropriate gravity setup. Returning to other methods, Wilson loop calculations are effectively a natural transform of the eigenspectrum, and might reveal features that are not apparent when looking at the low-lying spectrum. And potential calculations with static sources, if performed at larger N, might reveal something very interesting about the way the $N = \infty$ theory matches together the IF confining physics with the UV asymptotically free physics.

10. Conclusions

Large-N is at the intersection of much interesting theory and phenomenology. I have, in my talk, pointed to some of the questions that are doable and interesting – but there is more in the large number of topics that I have omitted. I would re-emphasise that many precise and informative calculations are possible with resources readily accessible to almost anyone – and really definitive calculations to the many lattice groups with Teraflop resources.

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